Chapter 10

Turbulent Viscosity Models

In this Chapter and the next we consider RANS models in which the Reynolds equations are solved for the mean velocity field. The Reynolds stresses—which appear as unknowns in the Reynolds equations—are determined by a turbulence model, either via the turbulent viscosity hypothesis or more directly from modelled Reynolds-stress transport equations (Chapter 11).

Turbulent viscosity models are based on the turbulent viscosity hypothesis, which was introduced in Chapter 4 and has been used in subsequent chapters. According to the hypothesis, the Reynolds stresses are given by

$$\langle u_i u_j \rangle = \tfrac{2}{3} k \delta_{ij} - \nu_T \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right), \tag{10.1}$$

or, in simple shear flow, the shear stress is given by

$$\langle uv \rangle = -\nu_T \frac{\partial \langle U \rangle}{\partial y}. \tag{10.2}$$

Given the turbulent viscosity field $\nu_T(\mathbf{x}, t)$, Eq. (10.1) provides a most convenient closure to the Reynolds equations, which then have the same form as the Navier-Stokes equations (Eq. 4.46 on page 96). It is unfortunate, therefore, that for many flows the accuracy of the hypothesis is poor. The deficiencies of the turbulent viscosity hypothesis—many of which have been mentioned above—are reviewed in Section 10.1.

If the turbulent viscosity hypothesis is accepted as an adequate approximation, all that remains is to determine an appropriate specification of the turbulent viscosity $\nu_T(\mathbf{x}, t)$. This can be written as the product of a velocity $u^*(\mathbf{x}, t)$ and a length $\ell^*(\mathbf{x}, t)$

$$\nu_T = u^* \ell^*, \tag{10.3}$$

and the task of specifying ν_T is generally approached through specifications of u^* and ℓ^*. In algebraic models (Section 10.2)—the mixing-length model, for example—ℓ^* is specified based on the geometry of the flow. In two-equation models (Section 10.4)—the k-ε model being the prime example—u^* and ℓ^* are related to k and ε for which modelled transport equations are solved.

10.1 Turbulent Viscosity Hypothesis

The turbulent viscosity hypothesis can be viewed in two parts. First, there is the *intrinsic* assumption that (at each point and time) the Reynolds-stress anisotropy $a_{ij} \equiv \langle u_i u_j \rangle - \frac{2}{3} k \delta_{ij}$ is determined by the mean velocity gradients $\partial \langle U_i \rangle / \partial x_j$. Second, there is the *specific* assumption that the relationship between a_{ij} and $\partial \langle U_i \rangle / \partial x_j$ is

$$\langle u_i u_j \rangle - \tfrac{2}{3} k \delta_{ij} = -\nu_T \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right), \tag{10.4}$$

or, equivalently,

$$a_{ij} = -2 \nu_T \bar{S}_{ij}, \tag{10.5}$$

where $\bar{S}_{ij}$ is the mean rate-of-strain tensor. This is, of course, directly analogous to the relation for the viscous stress in a Newtonian fluid:

$$-(\tau_{ij} + P \delta_{ij})/\rho = -2 \nu S_{ij}. \tag{10.6}$$

10.1.1 Intrinsic Assumption

To discuss the intrinsic assumption we first describe a simple flow in which it is entirely incorrect. Then it is shown that, in a crucial respect, the physics of turbulence is vastly different from the physics of the molecular processes that lead to the viscous stress law (Eq. 10.6). But finally, it is observed that for simple shear flows, the turbulent viscosity hypothesis is nevertheless quite reasonable.

Axisymmetric Contraction. Figure 10.1 is a sketch of a wind-tunnel experiment, first performed by Uberoi (1956), to study the effect on turbulence of an axisymmetric contraction. The air flows through the turbulence-generating grid into the first straight section, in which the mean velocity $\langle U_1 \rangle$ is (ideally) uniform. In this section there is no mean straining ($\bar{S}_{ij} = 0$), and the turbulence (which is almost isotropic) begins to decay.

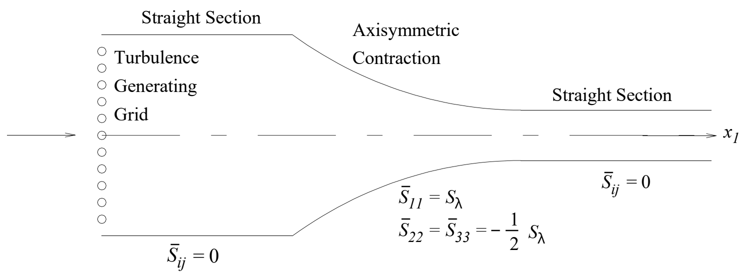

Figure 10.1: Sketch of an apparatus, similar to that used by Uberoi (1956) and Tucker (1970), to study the effect of axisymmetric mean straining on grid turbulence.

Following the first straight section there is an axisymmetric contraction which is designed to produce a uniform extensive axial strain rate, $\bar{S}_{11} = \partial\langle U_1\rangle/\partial x_1 = \mathcal{S}_\lambda$, and hence uniform compressive lateral strain rates, $\bar{S}_{22} = \bar{S}_{33} = -\frac{1}{2}\mathcal{S}_\lambda$. The quantity $\mathcal{S}_\lambda k/\varepsilon$ (evaluated at the beginning of the contraction) measures the mean strain rate relative to the turbulent time scale. Figure 10.2 shows measurements of the normalized anisotropies $(b_{ij} \equiv \langle u_i u_j\rangle/\langle u_k u_k\rangle - \frac{1}{3}\delta_{ij} = \frac{1}{2}a_{ij}/k)$ from the experiment of Tucker (1970) with $\mathcal{S}_\lambda k/\varepsilon = 2.1$. Also shown in Fig. 10.2 are DNS results for $\mathcal{S}_\lambda k/\varepsilon = 55.7$ obtained by Lee and Reynolds (1985). For this large value of $\mathcal{S}_\lambda k/\varepsilon$, rapid distortion theory (RDT, see Section 11.4) accurately describes the evolution of the Reynolds stresses. According to RDT, the Reynolds stresses are determined, not by the rate of strain, but by the total amount of mean strain experienced by the turbulence. In these circumstances the turbulence behaves, not like a viscous fluid, but more like an elastic solid (Crow 1968): the turbulent viscosity hypothesis is qualitatively incorrect.

In the experiment depicted in Fig. 10.1, following the contraction there is a second straight section. Since there is no mean straining in this section, the turbulent viscosity hypothesis inevitably predicts that the Reynolds-stress anisotropies are zero. But the experimental data of Warhaft (1980) show instead that the anisotropies generated in the contraction decay quite slowly, on the turbulent timescale k/ε (see Fig. 10.2). These persisting anisotropies exist, not because of the local mean strain rates (which are zero), but because of the prior history of straining to which the turbulence has been subjected.

Evidently, for this flow, both in the contraction section and in the downstream straight section, the intrinsic assumption of the turbulent viscosity

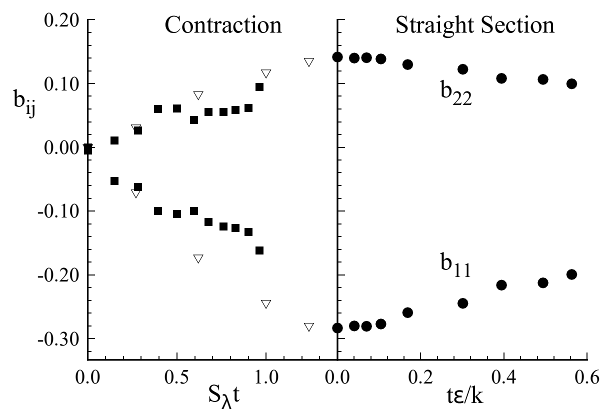

Figure 10.2: Reynolds-stress anisotropies during and after axisymmetric straining. Contraction: experimental data of Tucker (1970), $\mathcal{S}_\lambda k/\varepsilon = 2.1$; ∇ DNS data of Lee and Reynolds (1985), $\mathcal{S}_\lambda k/\varepsilon = 55.7$; flight time t from the beginning of the contraction is normalized by the mean strain rate $\mathcal{S}_\lambda$. Straight section: experimental data of Warhaft (1980); flight time from the beginning of the straight section is normalized by the turbulent timescale there.

hypothesis is incorrect: the Reynolds-stress anisotropies are not determined by the local mean rates of strain.

Comparison with Kinetic Theory. Simple kinetic theory for ideal gases (see, e.g., Vincenti and Kruger 1965, Chapman and Cowling 1970) yields the Newtonian viscous stress law (Eq. 10.6), with the kinematic viscosity given by

$$\nu \approx \tfrac{1}{2}\bar{C}\lambda, \tag{10.7}$$

where $\bar{C}$ is the mean molecular speed, and λ is the mean free path. It is natural to seek to justify the turbulent viscosity hypothesis through analogy with kinetic theory, and hence to give physical significance to u^* and ℓ^* by analogy to $\bar{C}$ and λ. But a simple examination of the different timescales involved shows that such an analogy has no general validity.

In simple laminar shear flow (with shear rate $\partial U_1/\partial x_2 = \mathcal{S} = \mathcal{U}/\mathcal{L}$), the ratio between the molecular timescale $\lambda/\bar{C}$ and the shear timescale $\mathcal{S}^{-1}$ is

$$\frac{\lambda}{\bar{C}}\mathcal{S} = \frac{\lambda}{\mathcal{L}}\frac{\mathcal{U}}{\bar{C}} \sim \mathrm{Kn}\mathrm{Ma}, \tag{10.8}$$

which is typically very small (e.g., 10^{-10}, see Exercise 10.1). The significance of the molecular timescale being relatively minute is that the statistical state of the molecular motion rapidly adjusts to the imposed straining. In contrast, for turbulent shear flows, the ratio of the turbulent timescale $\tau = k/\varepsilon$ to the mean shear timescale $\mathcal{S}^{-1}$ is not small: in the self-similar round jet $\mathcal{S}k/\varepsilon$ is of about 3 (Table 5.2 on page 135); in homogeneous turbulent shear flow experiments it is typically 6 (Table 5.4 on page 162); and in turbulence subjected to rapid distortions it can be orders of magnitude larger. Consequently, as already observed, turbulence does not adjust rapidly to imposed mean straining, and so (in contrast to the case of molecular motion) there is no general basis for a local relationship between stress and rate of strain.

Simple Shear Flows. The example of rapid axisymmetric distortion and the timescale considerations given above show that, in general, the turbulent viscosity hypothesis is incorrect. These general objections notwithstanding, there are important particular flows in which the hypothesis is more reasonable. In simple turbulent shear flows (e.g., the round jet, mixing layer, channel flow or boundary layer) the turbulence characteristics and mean velocity gradients change relatively slowly (following the mean flow). As a consequence, the local mean velocity gradients characterize the history of the mean distortion to which the turbulence has been subjected; and the Reynolds-stress balance is dominated by local processes—production, dissipation, pressure-rate-of-strain—with the non-local transport processes being small in comparison (see e.g., Figs. 7.35-7.38 on pages 324-325). In these circumstances, then, it is more reasonable to hypothesize that a relationship exists between the Reynolds stresses and the local mean velocity gradients.

An important observation is that in these particular flows (in which the turbulence characteristics change slowly following the mean flow) the production and dissipation of turbulent kinetic energy are approximately in balance, i.e., $\mathcal{P}/\varepsilon \approx 1$. In contrast, in the axisymmetric contraction experiment (Fig. 10.1), in the contraction section $\mathcal{P}/\varepsilon$ is much greater than unity, while in the downstream straight section $\mathcal{P}/\varepsilon$ is zero: in both of these cases the turbulent viscosity hypothesis is incorrect.

Gradient Diffusion Hypothesis. Related to the turbulent viscosity hypothesis is the gradient diffusion hypothesis

$$\langle \mathbf{u}\phi' \rangle = -\Gamma_T \nabla \langle \phi \rangle, \tag{10.9}$$

according to which the scalar flux $\langle \mathbf{u}\phi' \rangle$ is aligned with the mean scalar gradient (see Section 4.4). Most of the observations made above apply equally to the gradient diffusion hypothesis. In homogeneous shear flow it is found that the direction of the scalar flux is significantly different from that of the mean gradient (Tavoularis and Corrsin 1985). But in simple $2D$ turbulent shear flows (in the usual coordinate system) the scalar equation

$$\langle v\phi' \rangle = -\Gamma_T \frac{\partial \langle \phi \rangle}{\partial y}, \tag{10.10}$$

can be used to define Γ_T, and thus no assumption is involved (for this component). The turbulent Prandtl number σ_T can be used to relate ν_T and Γ_T, i.e., $\Gamma_T = \nu_T/\sigma_T$; and for simple shear flows, σ_T is of order unity (see, e.g., Fig. 5.34 on page 168).

Both ν_T and Γ_T can be written as the product of a velocity scale and a length scale (Eq. 10.3). They can also be expressed as the product of the square of a velocity scale and a time scale

$$\Gamma_T = u^{*2} T^*, \tag{10.11}$$

As shown in Section 12.4, in ideal circumstances, Γ_T can be related to statistics of the turbulence: u^* is the r.m.s. velocity u', and T^* is the Lagrangian integral timescale T_L (see Eq. 12.158 on page 514).

Exercise 10.1 According to simple kinetic theory (see, e.g., Vincenti and Kruger 1965) the kinematic viscosity of an ideal gas is

$$\nu \approx \tfrac{1}{2}\bar{C}\lambda, \tag{10.12}$$

and the mean molecular speed $\bar{C}$ is 1.35 times the speed of sound a. Show that the shear rate $\mathcal{S} = U/\mathcal{L}$ normalized by the molecular timescale $\lambda/\bar{C}$ is

$$\frac{\mathcal{S}\lambda}{\bar{C}} \approx 0.7\text{MaKn}, \tag{10.13}$$

where the Mach number and Knudsen number are defined by $\text{Ma} \equiv U/a$ and $\text{Kn} \equiv \lambda/\mathcal{L}$.

Use the relation $a^2 = \gamma p/\rho$ (with $\gamma = 1.4$) to show, that the ratio of the viscous shear stress τ_{12} to the normal stress (pressure) is

$$\frac{\tau_{12}}{P} \approx 0.9\text{MaKn}. \tag{10.14}$$

Using the values $a = 332 m/s$ and $\nu = 1.33 \times 10^{-5} m^2/s$ (corresponding to air at atmospheric conditions) and $\mathcal{S} = 1 s^{-1}$, obtain the following estimates:

$$\begin{aligned} \lambda &= 5.9 \times 10^{-8} m, \quad \lambda/\bar{C} = 1.3 \times 10^{-10} s, \\ \frac{\mathcal{S}\lambda}{\bar{C}} &= 1.3 \times 10^{-10}, \quad \frac{\tau_{12}}{P} = 1.7 \times 10^{-10}. \end{aligned} \tag{10.15}$$
